Resit Exam — Analysis (WPMA14004)

Tuesday 11 July 2017, 9.00h–12.00h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. Provide clear arguments for all your answers: only answering "yes", "no", or "42" is not sufficient. You may use all theorems and statements in the book, but you should clearly indicate which of them you are using.
3. The total score for all questions equals 90. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (5 + 5 + 5 = 15 points)

Let $A, B \subseteq \mathbb{R}$ be non-empty and assume that $a \leq b$ for all $a \in A$ and $b \in B$. Prove that:

(a) A is bounded above and B is bounded below;

(b) $\sup A \leq \inf B$;

(c) If $A \cup B = \mathbb{R}$ then $\sup A = \inf B$.

Problem 2 (3 + 3 + 3 + 6 = 15 points)

(a) Give the definition of a Cauchy sequence.

(b) Give an example of each of the following, or argue that such a request is impossible:

- (i) a Cauchy sequence with an unbounded subsequence;
- (ii) an unbounded sequence containing a subsequence that is Cauchy;
- (iii) a divergent monotone sequence with a Cauchy subsequence.

Problem 3 (3 + 7 + 5 = 15 points)

Consider the following set:

$$A = \left\{ \frac{1}{n} \; : \; n \in \mathbb{N} \right\}.$$

Prove the following statements:

(a) 0 is a limit point of A;

(b) If $x > 0$ or $x < 0$, then x is *not* a limit point of A;

(c) A is *not* compact, but $K = A \cup \{0\}$ *is* compact.

Problem 4 (5 + 5 + 5 = 15 points)

Let $\alpha \in \mathbb{R}$ and consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \ln\left(1 + \frac{1}{n}\right).$$

(a) Use the Mean Value Theorem to prove that

$$\frac{1}{n+1} < \ln(n+1) - \ln(n) < \frac{1}{n} \quad \text{for all} \quad n \in \mathbb{N}.$$

(b) Prove that the series converges for all $\alpha > 0$.

(c) Prove that the series diverges for all $\alpha \leq 0$.

Problem 5 (5 + 5 + 5 = 15 points)

Consider the following sequence of functions:

$$f_n : (0, \infty) \to \mathbb{R}, \qquad f_n(x) = \frac{nx}{1 + nx^2}.$$

(a) Compute the pointwise limit of (f_n) for all $x \in (0, \infty)$.

(b) Is the convergence uniform on $(0, \infty)$?

(c) Is the convergence uniform on $(1, \infty)$?

Problem 6 (10 + 5 = 15 points)

(a) Let $f : [a, b] \to \mathbb{R}$ be continuous. Prove that if $\int_a^b f = 0$ then there exists $c \in (a, b)$ such that $f(c) = 0$. Clearly state all the theorems that you are using!

(b) Give an example to show that the implication in part (a) is false if f is *not* continuous.

End of test (90 points)

Solution of Problem 1 (5 + 5 + 5 = 15 points)

(a) If $b_0 \in B$ is arbitrary, then it follows by assumption that $a \leq b_0$ for all $a \in A$. This shows that A is bounded above.
(2 points)

If $a_0 \in A$ is arbitrary, then it follows by assumption that $a_0 \leq b$ for all $b \in B$. This shows that B is bounded below.
(3 points)

(b) Any element $b_0 \in B$ is an upper bound of A. Hence, the least upper bound cannot be larger than that. Hence $\sup A \leq b_0$.
(2 points)

Since the element $b_0 \in B$ was chosen arbitrarily, it follows that $\sup A$ is a lower bound for B. The greatest lower bound of B cannot be smaller than that. Hence, $\sup A \leq \inf B$.
(3 points)

(c) Assume that $\sup A < \inf B$. Then there exists $x \in \mathbb{R}$ such that $\sup A < x < \inf B$. In particular, it follows that $x \notin A$ and $x \notin B$ so that $A \cup B \neq \mathbb{R}$. Equivalently, this means that if $A \cup B = \mathbb{R}$ then $\sup A = \inf B$.
(5 points)

Solution of Problem 2 (3 + 3 + 3 + 6 = 15 points)

(a) The sequence (a_n) is a Cauchy sequence if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$m, n \geq N \quad \Rightarrow \quad |a_n - a_m| < \epsilon.$$

(3 points)

(b) (i) A Cauchy sequence is bounded and hence every subsequence of a Cauchy sequence is bounded as well. This means that the request is impossible.
(3 points)

(ii) Consider, for example, the sequence $1, 0, 2, 0, 3, 0, 4, 0 \ldots$ This is an unbounded sequence, but the subsequence retaining only the terms with even index form a constant sequence (which is trivially a Cauchy sequence).
(3 points)

(iii) Without loss of generality we may assume that the sequence is monotonically increasing. If this sequence is divergent, then it follows that it must be unbounded (otherwise it would be convergent by the Monotone Convergence Theorem).
(3 points)

But if (x_n) is increasing and unbounded, then every subsequence is also unbounded. Indeed, since $n_k \geq k$ it follows that $x_{n_k} \geq x_k$ and for every $M > 0$ we can find $k \in \mathbb{N}$ such that $x_k > M$. Hence, no subsequence of (x_n) can be Cauchy.
(3 points)

Solution of Problem 3 (3 + 7 + 5 = 15 points)

(a) Let $\epsilon > 0$ be arbitrary and choose $n \in \mathbb{N}$ such that $1/n < \epsilon$. Then $1/n \in V_\epsilon(0) \cap A$, which shows that 0 is a limit point of A.
(3 points)

(b) Let $x < 0$ and pick $\epsilon < -\frac{1}{2}x$. Then $V_\epsilon(x) \cap A = \emptyset$ which means that x is not a limit point of A.
(2 points)

Let $x > 1$ and pick $\epsilon < \frac{1}{2}(x-1)$. Then $V_\epsilon(x) \cap A = \emptyset$ which means that x is not a limit point of A.
(2 points)

Let $0 < x \leq 1$ and pick $n \in \mathbb{N}$ such that

$$\frac{1}{n+1} < x \leq \frac{1}{n}$$

Then for $0 < \epsilon < 1/n - 1/(n+1)$ we have that $V_\epsilon(x) \cap A$ is either empty or it only contains the point $x = 1/n$. This shows that x is not a limit point of A.
(3 points)

(c) A is not closed since it does not contain all of its limit points. Hence, A is not compact.
(3 points)

$K = A \cup \{0\}$ is the closure of A and hence closed. Since K is bounded as well, it follows that K is compact.
(2 points)

Solution of Problem 4 (5 + 5 + 5 = 15 points)

(a) Let $n \in \mathbb{N}$ be arbitrary. The function $f(x) = \ln(x)$ is continuous on $[n, n+1]$ and differentiable on $(n, n+1)$. By the Mean Value Theorem there exists a point $c \in (n, n+1)$ such that

$$\frac{\ln(n+1) - \ln(n)}{(n+1) - n} = \ln'(c) = \frac{1}{c} \in \left(\frac{1}{n+1}, \frac{1}{n}\right),$$

which proves the desired inequality.
(5 points)

(b) By part (a) we have for all $n \in \mathbb{N}$ that

$$\frac{1}{n^\alpha} \ln\left(1 + \frac{1}{n}\right) = \frac{\ln(n+1) - \ln(n)}{n^\alpha} < \frac{1}{n^{\alpha+1}}.$$

Since the series $\sum_{n=1}^{\infty} 1/n^{\alpha+1}$ converges for all $\alpha > 0$ it follows by the Comparison Test that the given series converges as well.
(5 points)

(c) If $\alpha \leq 0$ then $1/n^\alpha \geq 1$ so that

$$\frac{1}{n^\alpha} \ln\left(1 + \frac{1}{n}\right) \geq \ln\left(1 + \frac{1}{n}\right) = \ln(n+1) - \ln(n) > \frac{1}{n+1}.$$

Since the series $\sum_{n=1}^{\infty} 1/(n+1)$ diverges, it follows by the Comparison Test and part (a) that the given series diverges as well.
(5 points)

Solution of Problem 5 (5 + 5 + 5 = 15 points)

(a) With $x > 0$ fixed it follows by the Algebraic Limit Theorem that

$$\lim f_n(x) = \lim \left(\frac{x}{1/n + x^2} \right) = \frac{x}{x^2} = \frac{1}{x}.$$

Hence, the pointwise limit is $f(x) = 1/x$.
(5 points)

(b) *Solution 1.* Note that

$$\sup_{x\in(0,\infty)} |f_n(x) - f(x)| = \sup_{x\in(0,\infty)} \frac{1}{(1+nx^2)x} = \infty \quad \text{for all} \quad n \in \mathbb{N},$$

which immediately shows that the convergence is not uniform on $(0, \infty)$.
(5 points)

Solution 2. Note that

$$|f_n(x) - f(x)| = \left| \frac{nx}{1+nx^2} - \frac{1}{x} \right| = \frac{1}{(1+nx^2)x} < \epsilon \quad \Leftrightarrow \quad n > \frac{1-\epsilon x}{\epsilon x^3}.$$

This shows that for a chosen $\epsilon > 0$ the corresponding N in the definition of convergence depends on x: a smaller x requires a bigger N. This shows that the convergence is not uniform on $(0, \infty)$.
(5 points)

(c) Note that

$$\sup_{x\in(1,\infty)} |f_n(x) - f(x)| = |f_n(1) - f(1)| = \frac{1}{1+n} \quad \text{for all} \quad n \in \mathbb{N},$$

which implies that

$$\lim_{n\to\infty} \left(\sup_{x\in(1,\infty)} |f_n(x) - f(x)| \right) = 0.$$

This shows that the convergence is uniform on $(1, \infty)$.
(5 points)

Solution of Problem 6 (10 + 5 = 15 points)

(a) Since f is continuous it is also integrable.
(2 points)

Hence, we can use the second part of the Fundamental Theorem of Calculus. Define $F(x) = \int_a^x f$. Since f is continuous in $[a, b]$ it follows that F is differentiable on $[a, b]$ and $F'(x) = f(x)$ for all $x \in [a, b]$.
(4 points)

By assumption, $F(a) = F(b) = 0$. Therefore, Rolle's Theorem (or the Mean Value Theorem) implies that $F'(c) = 0$ for some $c \in (a, b)$ which implies that $f(c) = 0$.
(4 points)

(b) Consider the function

$$f(x) = \begin{cases} -1 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Then f is integrable on $[-1, 1]$ and $\int_{-1}^1 f = 0$. However, $f(x) \neq 0$ for all $x \in [-1, 1]$.
(5 points)